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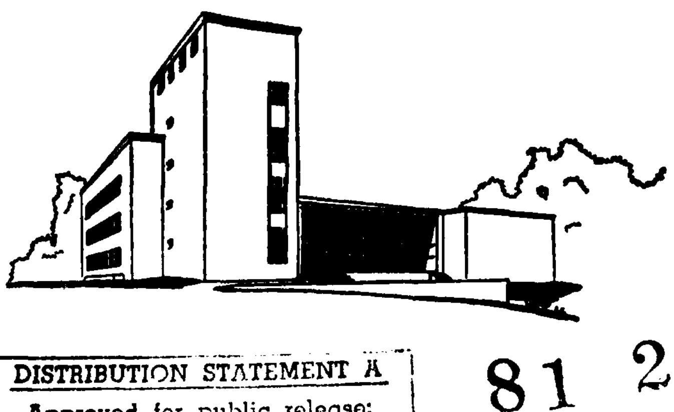

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THE TRAVELLING SALESMAN POLYTOPE AND {0,2} - MATCHINGS

by

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Abstract

A {0,2}-matching is an assignment of the integers 0,2 to the edges of a graph G such that for every node the sum of the integers on the incident edges is at most two. A tour is the 0-1-incidence vector of a hamilton cycle. We study the polytope P(G), defined to be the convex hull of the {0,2}-matchings and tours of G. When G has an odd number of nodes, the travelling salesman polytope, the convex hull of the tours, is a facet of P(G). We obtain the following results:

i) We completely characterize those facets of P(G) which can be induced by an inequality with 0-1-coefficients.

ii) We prove necessary properties for any other facet inducing inequality, and exhibit a class of such inequalities with the property that for any pair of consecutive positive integers, there exists an inequality in our class whose coefficients include these integers.

iii) We relate the facets of P(G) to the facets of the travelling salesman polytope. In particular, we show that for any facet F of the travelling salesman polytope, there is a unique facet of P(G) whose intersection with the travelling salesman polytope is exactly F.

1. Introduction

Let $G = (V,E)$ be a finite undirected graph and let $c = (c_j: j \in E)$ be a vector of real edge costs. The infamous travelling salesman problem is to find a hamilton cycle of G, the sum of whose edge costs is minimum. (If G has no hamilton cycle, this fact should be discovered.) A major obstacle to be overcome in this process is the verification of a proposed optimum tour. Indeed, even if one has discovered the optimal tour, but is forced to convince a nonbeliever of the tours optimality, it is generally necessary to perform a quantity of work effectively as large as that performed in finding the optimum tour in the first place.

This need for a good optimality condition prompted the development of the area of polyhedral combinatorial optimization. This approach was pioneered by Jack Edmonds in solving matching problems [3], matroid optimization problems [4], [5] and as a special case, branching problems [6]. The idea is to represent the feasible solutions to a discrete optimization problem by their incidence vectors and consider the convex hull of these vectors viewed as points of $\mathbb{R}^n$. If a linear system sufficient to define such a polyhedron can be discovered, then linear programming duality theory provides a general min-max optimality criterion.

So far, the results obtained using this approach have not been as successful for the travelling salesman problem as for these other problems. At present no complete characterization of a linear system sufficient to define the convex hull of the set of incidence vectors of the hamilton cycles -the so called travelling salesman polytope- is known. The first extensive study of this polytope was carried out by Grötschel [7] as a part of his doctoral dissertation. This continued earlier work of Chvátal [1] who introduced the class of " comb inequalities", which is at present the largest known class of essential inequalities for the travelling salesman polytope. The necessity of these inequalities was shown by Grötschel and Padberg [10]. Other results on the travelling salesman polytope have been obtained by Grötschel [8] and Maurras [13]. However, even though the incompleteness of these linear systems is unsatisfying from a theoretical point of view, these partial systems have provided the basis for successful cutting plane approaches to "real world" problems. See Grötschel [9], Padberg and Hong [14].

There is a problem that arises when dealing with the travelling salesman polytope that does not arise when dealing with the polyhedra of matroids and matchings: The travelling salesman polytope is not full dimensional. This means that there does not exist a unique (up to a positive multiple) minimal defining linear system as there does for these other polytopes. In fact an inequality can always be replaced by another obtained by multiplying by a positive constant and adding any linear combination of the equations which define the affine space containing the polytope. Generally, full dimensional polytopes seem more pleasant to handle, so what is often done when studying the travelling salesman polytope is to consider, in fact, the so called monotone travelling salesman polytope: the convex hull of the incidence vectors of the hamilton cycles
and all subsets of hamilton cycles of a graph. Then the travelling salesman polytope is a face of this larger polytope, the face obtained by requiring $\Sigma(x_j: j \in E) = |V|$.

We study here a different full dimensional extension of the travelling salesman polytope. If the number of nodes of G is odd, then, again, the travelling salesman polytope is a proper face of this larger polytope. In the next section we introduce this polytope and completely characterize all the essential inequalities of a defining linear system which can be scaled so as to have 0-1 coefficients. In Section 3 we prove several necessary properties of any non 0-1 essential inequalities, and give a class of such inequalities. These inequalities have the property that for G sufficiently large, any desired consecutive pair of positive integers can be obtained as coefficients. Finally, in Section 4, we relate the results of Section3 to the previous known classes of essential inequalities for the travelling salesman polytope. In particular, we show that there exists a natural injection of the set of facets of the travelling salesman polytope into the set of facets of our polytope.

One point of terminology should be clarified at this point. A facet of a polytope P is a maximal nonempty proper face, that is a face of dimension one less than that of P. A facet inducing inequality is any inequality which is satisfied by all members of P, and satisfied with equality by precisely the members of some facet F of P. For general polytopes, if we have a minimal defining linear system, then there will be exactly one facet inducing inequality for each facet of P. If the polytope is of full dimension, then every inequality in the system will be facet inducing. However if the polytope is not of full dimension, then this minimal defining system will also include sufficient linear equalities or inequalities to determine the affine space containing the polytope.

2. Tours and {0,2} - matchings

For any edge j of $G = (V,E)$ we let $\psi(i)$ denote the two nodes of G incident with j. For any $S \subseteq V$ we let $\delta(S)$ denote the set of edges having exactly one end in S and we let $\gamma(S)$ denote the set of edges having both ends in S. We abbreviate $\delta(\{v\})$ by $\delta(v)$ for $v \in V$. For any $J \subseteq E$ and any vector $x = (x_j: j \in E)$ we let $x(J) = \Sigma(x_j: j \in J)$. If K is any graph, we will sometimes use $E(K)$ and $V(K)$ to denote the edge set and nodeset respectively of K.

Now consider the following linear system:

(2.1) $\quad 0 \leq x_j \leq 1 \quad$ for all $j \in E$,

(2.2) $\quad x(\delta(i)) \leq 2 \quad$ for all $i \in V$,

(2.3) $\quad x(\gamma(S)) \leq |S| - 1 \quad$ for all $S \subset V$, $|S| \geq 3$.

We define a tour to be the incidence vector of the edges of a hamilton cycle of G. It is easily verified that the integer solutions to (2.1) - (2.3) are the tours of G and all incidence vectors of collections of node-disjoint paths. Moreover, if the inequalities in (2.2) are replaced by equations, then the integer solutions are precisely the tours. (This latter system is one of the earliest integer programming formulations of the travelling salesman problem (Dantzig, Fulkerson, Johnson [2])). The constraints(2.2) are called degree constraints; the constraints (2.3) are called subtour elimination constraints.

Now suppose we remove the upper bound from (2.1). That is, we replace it with

(2.4) $\quad 0 \leq x_j \quad$ for all $j \in E$.

The set of 0-1 valued solutions obviously remains unchanged but the set of integer solutions is greatly enlarged. A 1-matching of G is a set of edges meeting each node at most once. We say that it is perfect if it meets each node exactly once. Let M be a 1-matching of G and let $x = (x_j: j \in E)$ be defined by

$$x_j = \begin{cases} 0 & \text{if } j \in E-M \\ 2 & \text{if } j \in M. \end{cases}$$

We call such a vector a $\{0,2\}$ - matching of G. It is easily verified that any $\{0,2\}$ - matching x satisfies (2.2) - (2.4). Conversely, if we let $P(G)$ denote the convex hull of the tours and $\{0,2\}$ - matchings of G, then $P(G)$ is the covex hull of the integer solutions to (2.2) - (2.4). This follows from the observation that any integer solution to (2.2)-(2.4) other than a tour or a $\{0,2\}$-matching can be written as a convex combination $.5\,x_1 + .5\,x_2$ where x_1 and x_2 are $\{0,2\}$-matchings.

In the case that $|V|$ is even, any tour can also be expressed as $.5\,x_1 + .5\,x_2$, chosing x_1 and x_2 as the two complementary perfect $\{0,2\}$-matchings contained in the edges of the tour. Thus when $|V|$ is even the vertices of $P(G)$ are just the $\{0,2\}$-matchings. However, when $|V|$ is odd, the situation is quite different. Let $TSP(G)$ denote the travelling salesman polytope of G, i.e., the convex hull of the set of tours of G. Then $TSP(G) \subset P(G)$ and if $|V|$ is odd, then $TSP(G)$ is the face of $P(G)$ obtained by taking the intersection of $P(G)$ with the affine space defined by

$$(2.5) \qquad x(\delta(i)) = 2 \qquad \text{for all } i \in V.$$

This is because a graph with an odd number of nodes cannot have a perfect 1-matching and therefore if x is a $\{0,2\}$ -matching of G, then there must exist at least one node v for which $x(\delta(v)) < 2$. Conversely, every tour of G satisfies (2.5). Therefore, our objective in this section is to determine several classes of facets of $P(G)$ which we will then relate to $TSP(G)$.

For any $S \subseteq V$ we let $G[S]$ denote the node induced subgraph of G induced by S. We say that $S \subseteq V$ is hypomatchable (or 1-critical) if for every $v \in S$, the graph $G[S - \{v\}]$ has a perfect 1-matching. Necessarily, this implies that $|S|$ is odd. Let $Q = \{S \subseteq V\colon S \text{ is hypomatchable}\}$ and let $M(G)$ be the convex hull of the incidence vectors of the 1-matchings of G. Edmonds [3] proved the following:

Theorem 2.1 $\quad M(G) = \{x \in \mathbb{R}^E$:

$$(2.6) \qquad x_j \geq 0 \qquad \text{for all } j \in E,$$
$$(2.7) \qquad x(\delta(i)) \leq 1 \quad \text{for all } i \in V,$$
$$(2.8) \qquad x(\gamma(S)) \leq (|S| - 1)/2 \quad \text{for all } S \in Q\}.$$

(In fact, the theorem as stated by Edmonds had Q equal to the set of all odd cardinality subsets of V. However, the restriction to hypomatchable sets is implicit in his algorithm used to prove the theorem.) This system of inequalities is "almost" minimal. Pulleyblank and Edmonds [15] showed that all the inequalities (2.6) are necessary, all the inequalities (2.7) which do not violate a rather technical condition are necessary and an inequality (2.8) is necessary if and only if $G[S]$ is noseparable, i.e., contains no cut node.

Since a vector x is a $\{0,2\}$ - matching if and only if $x/2$ is the incidence vector of a 1-matching, a linear system sufficient to define the convex hull of the set of 2-matchings of G can be obtained by simply doubling the right hand sides of the linear system (2.6) - (2.8), and trivially, this linear system defines $P(G)$ for $|V|$ even. But when $|V|$ is odd, there is an inequality of the form (2.8) which requires $x(E) \leq |V| - 1$, and of course, every tour of G violates this inequality.

Since $P(G)$ contains all $\{0,2\}$-matchings of G, $P(G)$ is of full dimension Therefore for each facet F of $P(G)$ there exists a unique (up to a positive multiple) inequality $ax \leq \alpha$ such that $F = \{x \in P(G) : ax = \alpha\}$ and every $x \in P(G)$ satisfies as $\leq \alpha$. Moreover, the set of all such inequalities is the minimal defining linear system which we would like to find. Unfortunately, we are unable to explicitly describe this system, but in the following three propositions we define three classes of such facet-inducing inequalities. We will then show that every facet-inducing inequality with 0-1 coefficients belongs to one of these classes.

<u>Proposition 2.2</u> *For every $j \in E$, $x_j \geq 0$ induces a facet of $P(G)$.*

<u>Proof.</u> Let $\bar{0}$ denote the zero vector indexed by E and let u^k for $k \in E$ denote the vector which is zero everywhere but the k^{th} coordinate and $u_k^k = 2$. Then $\{\bar{0}\} \cup \{u^k : k \in E - \{j\}\}$ is a set of $|E|$ affinely independent vectors satifying $x_j = 0$. Since $\{x \in P(G) : x_j = 0\}$ is a proper face of $P(G)$, the dimension of this face is $|E| - 1$ and the result follows. □

It is clear that those graphs G which have isolated nodes are uninteresting from a point of view of $P(G)$, since their deletion leaves the polytope unchanged. Henceforth we will always assume that G has no isolated nodes, however, G need not be connected. Of course, if G is not connected, then there exist no tours so the result really reduce to results on $M(G)$.

Proposition 2.3 *For every $v \in V$, $x(\delta(v)) \leq 2$ does not induce a facet of $P(G)$ if and only if v has a single neighbor $w \in V$ and $\delta(w) \nsubseteq \delta(v)$.*

Proof. If v has a single neighbor u then any $x \in P(G)$ satisfying $x(\delta(v)) = 2$ also satisfies $x(\delta(u)) = 2$. If there exists $k \in \delta(u) - \delta(v)$ then the unit vector u^k defined in the proof of Proposition 2.2 satisfies $u^k(\delta(u)) = 2$ but $u^k(\delta(v)) = 0$. Therefore $\{x \in P(G): x(\delta(v)) = 2\}$ is not a maximal proper face of $P(G)$ and so $x(\delta(v)) = 2$ does not induce a facet.

Conversely, suppose that v has a single neighbor w but $\delta(w) = \delta(v)$. Let $h \in \delta(v)$. For any $j \in E - \delta(v)$ let $\tilde{u}^j$ be defined by

$$\tilde{u}^j_k \equiv \begin{cases} 0 & \text{if } k \notin \{h,j\} \\ 2 & \text{if } k \in \{h,j\}. \end{cases}$$

Then $\{\tilde{u}^j: j \in E - \delta(v)\} \cup \{u^k: k \in \delta(v)\}$ is a set of $|E|$ affinely independent $\{0,2\}$-matchings of G, all satisfying $x(\delta(v)) = 2$. Therefore $x(\delta(v)) \leq 2$ induces a facet of $P(G)$.

Finally suppose that v has more than one neighbor. Then for any $j \in E - \delta(v)$ there exists a $\{0,2\}$-matching $\bar{u}^j$ which is zero everywhere except for the j^{th} component and one component corresponding to a member of $\delta(v)$. Then, as before, $\{\bar{u}^j: j \in E - \delta(v)\} \cup \{u^k: k \in \delta(v)\}$ is a set of $|E|$ affinely independent $\{0,2\}$-matchings of G satisfying $x(\delta(v)) = 2$, so $x(\delta(v)) \leq 2$ induces a facet of $P(G)$.

[]

In fact, the preceding two propositions and the following one follow immediately from the facet characterizations [1] of the matching polytope $M(G)$. For suppose that $ax \leq \alpha$ is a facet inducing inequality for $M(G)$ and that $ax \leq 2\alpha$ is a valid inequality for $P(G)$. Then there is a set M of $|E|$ affinely independent incidence vectors x of 1-matchings all satisfying $ax = \alpha$. The set $\bar{M} \equiv (2 \cdot x : x \in M)$ is then a set of $|E|$ affinely independent $\{0,2\}$-matchings all satisfying $ax = 2\alpha$, which establishes that the inequality is facet inducing for $P(G)$.

If $|V|$ is odd, then, of course, no perfect 1-matching or $\{0,2\}$-matching of G can exist. Thus we define an np("near perfect") 1-matching to be a 1-matching which contains an edge incident with every node of V but one. Similarly, an np-$\{0,2\}$-matching is a $\{0,2\}$-matching x of G satisfying $x(E) = |V| - 1$. In other words, only one node is unsaturated. Then a graph G is hypomatchable if and only if for every $v \in V$, there exists an np-$\{0,2\}$-matching (or a np-1-matching) which leaves v unsaturated. In [15], the following theorem was proved.

Theorem 2.4 *If G is a nonseparable hypomatchable graph, then there exist $|E|$ np-1-matchings of G, whose incidence vectors are affinely independent.*

This result was proved constructively, via an algorithm which actually constructed the np-1-matchings. Using this result, it was then shown that for $S \subseteq V$ such that $|S| \geq 3$, $G[S]$ hypomatchable and nonseparable, the inequality $x(\gamma(S)) \leq (|S| - 1)/2$ induces a facet of $M(G)$. A shorter, nonconstructive proof of this result has been obtained by Lovász, which we describe here.

Lemma 2.5 *For every $S \subseteq V$ such that $|S| \geq 3$ and $G[S]$ is hypomatchable and nonseparable, $x(\gamma(S)) \leq (|S| - 1)/2$ induces a facet of $M(G)$.*

Proof. (Lovász). Let X be the set of incidence vectors x of 1-matchings of G which satisfy $x(\gamma(S)) = (|S| - 1)/2$. Since the inequality $x(\gamma(S)) \leq (|S| - 1)/2$ is easily seen to be satisfied by all members of $M(G)$, all we need show is that the affine rank of X is equal to $|E|$, or in other words, there is a unique (up to a positive multiple) nonzero vector $a = (a_j : j \in E)$ and scalar α such that $ax = \alpha$ for every $x \in X$. To do this, we will show that any such a must satisfy $a_j = k$ for some constant k, for all $j \in \gamma(S)$ and $a_j = 0$ for all $j \in E - \gamma(S)$. For then if we "scale" a by dividing every component by k we see that this inequality must be a scalar multiple of the inequality $x(\gamma(S)) \leq (|S| - 1)/2$.

So suppose there exists $i \in S$ such that a_j takes on different values for edges in $\delta(i) \cap \gamma(S)$. Let the graph G' be obtained from $G[S]$ by "splitting" i into two nodes i' and i'' such that all the edges j of $\delta(i) \cap \gamma(S)$ for which a_j takes on the minimum value are incident with i' and all the others are adjacent with i'' . Since $G[S]$ was nonseparable, G' is connected and in addition must have a perfect 1-matching. For if not, by Tutte's theorem [16], there would exist a nonempty subset Y of the nodes of G' such that deleting Y creates more than $|Y|$ odd cardinality components. But then it is easily verified that for any node $v \in Y$, if we let v' be the node of G corresponding to v , the graph $G[S-\{v'\}]$ does not have a perfect 1-matching contrary to $G[S]$ being hypomatchable. So let x^* be the incidence vector of a perfect matching of G', and let $j' \in \delta(i')$ and $j'' \in \delta(i'')$ be such that $x^*_{j'} = x^*_{j''} = 1$. Let x^1 and x^2 be obtained from x^* by setting the j' and j'' components respectively to zero. Then x^1, $x^2 \in X$ but $ax^1 > ax^2$, a contradiction to $ax = \alpha$ for all $x \in X$. Therefore, the value of a_j is constant for all $j \in \gamma(S)$. Moreover, it is easily verified that for every $j \in E - \gamma(S)$, there exists $x' \in X$ such that $x'_j = 1$. Moreover, the vector x'' obtained from x' by setting the j-th component to 0 also belongs to X . Therefore, we must have $a_j = 0$ for $j \in E - \gamma(S)$ and the result follows. □

<u>Proposition 2.6.</u> *For every $S \in V$ such that $|S| \geq 3$ and $G[S]$ is hypomatchable and nonseparable, $x(\gamma(S)) \leq (|S| - 1)$ induces a facet of $P(G)$.*

<u>Proof.</u> This is an immediate corollary of Lemma 2.5. □

The important difference between Lemma 2.5 and Proposition 2.6 is that in the latter we were forced to restrict S to being a <u>proper</u> subset of V , because every tour of G violates the inequality $x(E) \leq |V| - 1$. For the case $S = V$, we have the following result for $P(G)$.

<u>Proposition 2.7.</u> *Let $G' = (V,E')$ be a spanning subgraph of G which is hypomatchable , nonseparable and nonhamiltonian and such that E' is maximal with this property. Then $x(E') \leq |V| - 1$ induces a facet of $P(G)$.*

<u>Proof.</u> Since G' is nonhamiltonian and $|V|$ is odd, every member of $P(G)$ must satisfy $x(E') \leq |V| - 1$. All we need show is that there exist $|E|$ affinely independent members of $P(G)$, all of which satisfy $x(E') = |V| - 1$. First we note that since G' is nonseparable and hypomatchable, it follows from Proposition 2.4 that there exists a set X of $|E'|$ affinely independent incidence vectors of np-1-matchings of G' . Let $\overline{X}$ be obtained from X by taking each $x \in X$, doubling it and defining the j-th component to be zero for all $j \in E - E'$. Then $\overline{X}$ is a set of $|E'|$ affinely independent np-{0,2}-matchings of G . Moreover, $x_j = 0$ for all $j \in E - E'$, for all $x \in \overline{X}$. For each $j \in E - E'$, there exists a hamilton cycle whose edges are contained in $E' \cup \{j\}$, by the maximality of E' . Let t^j be the tour cor-

responding to such a hamilton cycle. Then $t^j(E') = |V| - 1$, for all $j \in E - E'$ and we let $T = \{t^j : j \in E - E'\}$. It is easily seen that $T \cup \overline{X}$ is a set of $|E|$ affinely independent members of $P(G)$ all satisfying $x(E') = |V| - 1$, since for $j \in E - E'$, t^j is the only member of $T \cup \overline{X}$ for which the j-th component is nonzero. []

We note that if G is nonhamiltonian, then the inequality of the previous proposition is simply $x(E) \leq |V| - 1$ which is facet inducing for $P(G)$ if and only if G is nonseparable and hypomatchable. However, when G is hamiltonian, then G' must be a proper spanning subgraph of G, which is therefore not node induced. In general, the number of these subgraphs is very large.

We make use of one more preliminary result. For any $X \subseteq V$ let $c(X)$ denote the number of components of $G[V-X]$ having an odd number of nodes. Tutte's classical theorem characterizing those graphs having perfect 1-matchings is the following:

Theorem 2.8 (Tutte [16]). *G has a perfect 1-matching if and only if for every $X \subseteq V$, $|X| \geq c(X)$.*

A less classical theorem characterizing those graphs which are hypomatchable was proved independently by Pulleyblank and Edmonds [15] and Lovâsz [11].

Theorem 2.9. *$G = (V,E)$ is hypomatchable if and only if $|V|$ is odd and for every nonempty $X \subset V$, $|X| \geq c(X)$.*

Of course, the important part of this theorem is the sufficiency of the condition, i.e. the assertion that if G is not hypomatchable and $|V|$ is odd then there exists nonempty $X \subseteq V$ such that $|X| < c(X)$. It is not difficult to strengthen this in the following manner.

Corollary 2.10. *Let $G = (V,E)$ be a nonhypomatchable graph with $|V|$ odd. Then there exists nonempty $X^* \subseteq V$ which maximizes $c(X) - |X|$ over all nonempty $X \subseteq V$, and such that $G[V-X^*]$ consists only of (at least $|X^*| + 1$) hypomatchable components.*

Proof. We prove by induction on $|V|$. If $|V| = 1$ then G is hypomatchable; if $|V| = 3$ then the assertion is easily checked. Suppose G has k nodes and the result is true for all smaller graphs. By Theorem 2.9 there exists nonempty $X \subseteq V$ such that $c(X) - |X| > 0$, let X^* be chosen such that $c(X^*) - |X^*|$ is maximum and, subject to this, the number of nonhypomatchable components of $G[V-X^*]$ is minimum. If there are no such components, then we are done, so suppose that S is the nodeset of a nonhypomatchable component. If $|S|$ is odd, then by induction there is $\phi \neq X_S \subseteq S$ such that $G[S-X_S]$ consists of at least $|X_S|+1$ hypomatchable components. Then

$$c(X^* \cup X_S) - |X^* \cup X_S| \geq c(X^*) - |X^*| \quad ,$$

but $G[V-(X^* \cup X_S)]$ contains fewer nonhypomatchable components than does $G[V-X^*]$, a contradiction. If $|S|$ is even then let v be any node of S, which is not a cutnode of $G[S]$ and let $X' \equiv X \cup \{v\}$. Then

$$c(X') - |X'| = c(X^*) - |X^*|$$

and if $G[S-\{v\}]$ is hypomatchable, then we have contradicted the choice of X^*. If not, then as before we use induction to find $X_S \subseteq S-\{v\}$ such that $X' \cup X_S$ contradicts the choice of X. []

Our next theorem provides a characterization of all those facet inducing inequalties of $P(G)$ which can be scaled so as to have $0-1$ coefficients. Thus we say that an inequality $ax \leq \alpha$ is a <u>$0-1$-inequality</u> if every $a_j \in \{0, -1, 1\}$. We will also say that such an inequality is a <u>$0-k$-inequality</u> if every $a_j \in \{0, -k, k\}$ for some positive real number k. Then, of course, to any $0-k$-inequality there corresponds a (unique) $0-1$-inequality obtained by multiplying by $1/k$.

Note that this definition allows us to consider the inequality $x_j \geq 0$ for $j \in E$ (equivalently, $-x_j \leq 0$) as a $0-1$-inequality. It might be asked whether there exist other facet inducing inequalities $ax \leq \alpha$ for $P(G)$ having $a_j < 0$ for some j. We can answer this in the negative; all others are obtained from nonnegativity constraints by scaling.

<u>Lemma 2.11.</u> *If $ax \leq \alpha$ is a facet inducing inequality for $P(G)$ having $a_j < 0$ for some $j \in J$, then this inequality must be $a_j x_j \leq 0$.*

<u>Proof.</u> Since $P(G)$ is of full dimension, if we let M be the set of $\{0,2\}$-matchings x of G satisfying $ax = \alpha$ and let T be the set of tours t of G satisfying $at = \alpha$ then the affine rank of $X \equiv M \cup T$ must be $|E|$. Therefore $ax = \alpha$ is the unique hyperplane containing all elements of X. Suppose $a_j < 0$. If there existed $\hat{x} \in M$ for which $\hat{x}_j > 0$, then setting the j-th component of $\hat{x}$ to 0 would yield another $\{0,2\}$-matching of G violating $ax \leq \alpha$. If there existed t with $t_j > 0$, then setting the j-th component to zero gives the incidence vector $\bar{x}$ of a hamilton path of G, for which

$a\bar{x} > \alpha$. But $\bar{x}$ is the average of two $\{0,2\}$-matchings of G, at least one of which must violate $ax \leq \alpha$. Thus we must have $x_j = 0$ for all $x \in X$ so $ax \leq \alpha$ is a positive multiple of the nonnegativity constraint $-x_j \leq 0$.

[]

For any $J \subseteq E$ we let $r(J)$ denote the maximum possible value for $x(J)$ for all $\{0,2\}$-matchings and tours x of G. This "rank" function is important for it is the smallest possible value for α if the 0-1-inequality $x(J) \leq \alpha$ is to be valid for $P(G)$. Moreover, if $\alpha > r(J)$, then no member of $P(G)$ can satisfy $x(J) = \alpha$. Thus $r(J)$ is the only possible value for α if $x(J) \leq \alpha$ is to induce a facet of $P(G)$.

Finally, let W denote the set of all $v \in V$ such that either v has at least two neighbors, or, if v has a single neighbor w, then $\delta(v) = \delta(w)$.

Theorem 2.12 *The following is the complete set of facet inducing 0-1-inequalities of $P(G)$:*

(2.9)	$x_j \geq 0$	*for all* $j \in E$
(2.10)	$x(\delta(i)) \leq 2$	*for all* $i \in W$
(2.11)	$x(\gamma(S)) \leq \|S\| - 1$	*for all* $S \subsetneq V$, $\|S\| \geq 3$, $G[S]$ *nonseparable, hypomatchable*
(2.12)	$x(E') \leq \|V\| - 1$	*for all edge maximal spanning subgraphs* $G'=(V,E')$ *of* G *which are hypomatchable, nonhamiltonian and nonseparable.*

Proof. We saw in Propositions 2.2,2.3,2.6 and 2.7 that all these inequalities do induce facets of P(G). Now we show that every facet inducing 0-1-inequality is of one of the above types. Let $ax \leq \alpha$ be facet inducing. By Lemma 2.11 if $ax \geq \alpha$ is not of the form (2.9) we must have $a \geq 0$, so let $E' \equiv \{j \in E: a_j = 1\}$. Then the inequality $ax \leq \alpha$ must be $x(E') \leq r(E')$. Jack Edmonds observed, in the context of matroid polyhedra, that if such an inequality is facet inducing, then two properties must hold: First, E' must be closed, i.e., for every $j \in E - E'$, we must have $r(E' \cup \{j\}) > r(E')$. Otherwise $x(E' \cup \{j\}) \leq r(E' \cup \{j\}) = r(E')$ would be a stronger valid inequality than $ax \leq \alpha$, contradicting the necessity of a facet-inducing inequality. Second, E' must be nonseparable, i.e. there cannot exist nonempty $S,T \subseteq E'$ such that $S \cup T = E'$ and $r(S) + r(T) = r(E')$. For in this case, the inequality $x(E') \leq r(E')$ is implied by the sum of the inequalities $x(S) \leq r(S)$ and $x(T) \leq r(T)$ which means that it can be replaced with these two different inequalities, again contradicting the necessity. So we know that E' is closed and nonseparable.

Let V' be the set of nodes incident with edges in E'. If the graph $G' = (V',E')$ has a perfect $\{0,2\}$-matching or contains a hamilton cycle of G, then $r(E') = |V'|$ and so $ax \leq \alpha$ is implied by one half of the sum of the degree constraints for the nodes in V'. Since the degree constraints are valid inequalities the facet $ax \leq \alpha$ can be necessary only if it is identical to a degree constraint (2.10). Furthermore the assumption that G' has a perfect $\{0,2\}$- matching means that $V' = \{u,v\}$ and $\delta(u) = \delta(v) = E'$.

Now suppose that G' is hypomatchable. Then $r(E') = |V'| - 1$. If $V' \subsetneq V$, then since E' is closed we must have $E' = \gamma(V')$. If G' were separable, then E' would be separable, so we must have $G' = G[V']$ is hypomatchable, nonseparable, so that $ax \leq \alpha$ is an inequality of the form (2.11). If $V' = V$, then G' is a hypomatchable non-hamiltonian spanning subgraph of G which must also be nonseparable and edge maximal with these properties, since E' is closed and nonseparable. Thus $ax \leq \alpha$ is a constraint of the form (2.12).

Next, suppose that G' is not hypomatchable, but $|V'|$ is odd. By Corollary 2.10 there exists nonempty $X \subseteq V'$ such that $G'[V' - X]$ consists of at least $|X| + 1$ hypomatchable components, and $c'(X) - |X|$ is maximized, where $c'(X)$ denotes the number of odd components of $G'[V' - X]$. If we sum the degree constraints (2.10) for the nodes of X and the constraints (2.11) for the node sets of the components of $G'[V' - X]$ (or the nonseparable blocks of these components if they contain cutnodes) then we obtain a valid inequality $x(\bar{E}) \leq |V'| - (c'(X) - |X|)$, where $\bar{E} \supseteq E'$.(If some of these components are single nodes, the constraint (2.11) is trivial and can be dropped.)We will show that $r(E') = |V'| - (c'(X) - |X|)$ which will contadict $x(E') \leq r(E')$ being a facet, since we can obtain it (or a stronger inequality) from other inequalities.

Clearly $r(E') \leq |V'| - (c'(X) - |X|)$; all we need do is find some $x^* \in P(G)$ giving equality. Construct a bipartite graph $\tilde{G}$ from G' having one node $v(x)$ for each $x \in X$, one node $v(K)$ for each component K of $G'[V' - X]$ and an edge joining $v(x)$ and $v(K)$ if and only if x was adjacent (in G') to some node of K. If there is no 1-matching which covers all nodes $v(x)$ for $x \in X$, then by Hall's theorem, there is a set

$\tilde{X} \subseteq X$ such that fewer than $|\tilde{X}|$ nodes $v(K)$ for components K of $G'[V' - X]$ are adjacent to nodes $v(x)$ for $x \in \tilde{X}$. But then $c'(X - \tilde{X}) - |X - \tilde{X}| > c'(X) - |X|$ a contradiction. So we can construct x by letting $x_j^* = 2$ for each edge corresponding to an edge of a maximum 1-matching of $\tilde{G}$, x_j^* be defined equal to an appropriate np-{0,2}-matching for each component of $G'[V' - X]$ and $x_j^* = 0$ otherwise. Then $x^* \in P(G)$ and $x^*(E') = 2|X| + \Sigma\{|S| - 1 : S$ is the nodeset of a component of $G'[V' - X]\ \} = |V'| - (c'(X) - |X|)$.

The final case to consider is that G' is not hypomatchable, and $|V'|$ is even. Since we assume that G' does not have a perfect {0,2}-matching we can now use a proof that parallels that of the last two paragraphs. For any $v \in V'$ let $V'' = V' - \{v\}$ and $c^v(X)$ be the number of odd components of $G'[V''-X]$. Consider the maximum value of $c^v(X) - |X|$ over all $v \in V'$ and all nonempty $X \subseteq V''$. To avoid cumbersome notation we denote by v and X an optimal solution. By Corollary 2.10 we know that X can be chosen so that every component of $G'[V'' - X]$ is hypomatchable, since $|V''|$ is odd. Let $X' = X \cup \{v\}$. As earlier, by summing the degree constraints for the nodes of X' and the constraints (2.11) for the components of $G'[V'' - X]$ we get the valid inequality $x(\bar{E}) \leq |V'| - (c^v(X) - |X'|)$. Again this inequality implies $x(E') \leq r(E')$ if there exists a matching $x^* \in P(G)$ such that $x^*(E') = |V'| - (c^v(X) - |X'|)$. Now consider the bipartite graph G with a node $n(x)$ for each $x \in X'$, a node $n(K)$ for each component K of $G'[V'' - X]$ and an edge joining $n(x)$ to $n(K)$ if x is adjacent (in G') to some node of K. If there is no 1-matching which covers every node $n(x)$ of $\tilde{G}$ then, by Hall's theorem, there is a set $\tilde{X} \subseteq X'$ such that fewer than $|\tilde{X}|$ nodes $v(K)$ are adjacent to $\tilde{X}$.

If $v \notin \tilde{X}$, then $c^v(X - \tilde{X}) - |X - \tilde{X}| > c^v(X) - |X|$ a contradiction. So $v \in X$. Let $u \in X' - \tilde{X}$ and $\hat{X} = (X' - \{u\}) - (\tilde{X} - \{v\})$. Now $c^u(\hat{X}) - |\hat{X}| > c^v(X) - |X|$ which is again a contradiction. So $\tilde{G}$ has a matching that saturates X'. As earlier this matching can be utilized to construct a {0,2}-matching x^* of $G'(V')$ which satisfies $x^*(E') = |V'| - (c^v(X) - |X'|)$.

$\square$

3. General Facets of $P(G)$

We discuss three topics in this section. First we establish some necessary conditions which must be satisfied by any facet inducing non-0-1 inequality of $P(G)$. Second, we describe such a class of non-0-1 inequalities having the following property : For any pair $(s,s+1)$ of consecutive positive integers, there exists a graph G and a facet inducing inequality whose coefficients include s and $s+1$. Third we describe a lifting procedure which allows us to obtain facets of $P(G)$ from facets of $P(G')$ for a subgraph G' of G.

Let $ax \leq \alpha$ be an integer inequality. We say that this inequality is non-(0-1) if it is impossible to scale the coefficients so that $a_j \in \{0,\pm 1\}$ for all $j \in E$. In other words, there exist two nonzero coefficients with different magnitudes. If t is a tour of $G = (V,E)$, we let $E(t) = \{j \in E : t_j = 1\}$.

Theorem 3.1 *Let* $ax \leq \alpha$ *be a non-0-1 facet inducing inequality for* $P(G)$. *Let* $E^+ \equiv \{j \in E : a_j \neq 0\}$ *and let* G^+ *be the subgraph of* G *induced by the edges in* E^+. *Let* M *be the set of* $\{0,1\}$*-matchings* x *satisfying* $ax = \alpha$ *and let* T *be the set of tours* t *satisfying* $at = \alpha$. *Then*

(3.1) $a_j > 0$ *for all* $j \in E^+$ *and* $\alpha > 0$;

(3.2) *every* $x \in M$ *is a np-matching of* G;

(3.3) M *contains a np-matching deficient at every node of* G;

(3.4) *any basis of* $T \cup M$ *contains at least one tour* t *for which* $E(t) \subseteq E^+$; *there exists a basis* B *of* $T \cup M$ *such that every* $t \in B$ *satisfies* $|E(t) - E^+| \leq 1$.

Note that (3.4) implies that G^+ is a spanning, hamiltonian subgraph of G, which of course implies that G^+ is hypomatchable. Condition (3.3) adds that M contains a np-matching of G deficient at every node of G. Moreover, there exists a basis of $M \cup T$, thus, a subset of the points sufficient to uniquely define the facet, consisting solely of np-matchings and tours which are either contained in G^+ or else induce hamilton paths in G^+.

Proof. Since $P(G)$ is of full dimension, the affine rank of $X \equiv M \cup T$ must be $|E|$, so $ax = \alpha$ is the unique hyperplane containing all elements of X. Since $ax \leq \alpha$ is a non-0-1-inequality, Lemma 2.11 yields $a \geq 0$, which in turn implies $a > 0$, so (3.1) is immediate. Now we show that G^+ must be connected. If not, let K be a component of G^+ and let a' and a'' be defined by

$$a_j' \equiv \begin{cases} a_i & \text{if } j \in E(K) \\ 0 & \text{otherwise} \end{cases}$$

$$a_j'' \equiv \begin{cases} 0 & \text{if } j \in E(K) \\ a_i & \text{otherwise} \end{cases}$$

If there exists $x^1, x^2 \in X$ such that $a'x^1 > a'x^2$, then we can find such $x^1, x^2 \in M$. For let $t \in T$. Since we assume G^+ is not connected, $E(t) \cap E^+$ will consist of some number of disjoint paths, and consequently can be expressed as the average of two $\{0,2\}$-matchings x^1, x^2 both of which must be in M. But then we must have $a'x^1 \leq a't \leq a'x^2$ and so one of x^1, x^2 would serve as a substitute for t. Since, therefore, $x^1, x^2 \in M$ we can define x^* equal to x^1 on $E(K)$ and equal to x^2 on the rest of G and then $ax^* > \alpha$, a contradiction. Therefore

(3.5) G^+ is connected.

If every $x \in M$ satisfied $x(\delta(v)) = 2$ for some $v \in V$, then since every $t \in T$ must satisfy $t(\delta(v)) = 2$, we must have $ax \leq \alpha$ being a degree constraint (2.10). Since we have assumed that $ax \leq \alpha$ does not induce a 0-1 facet, we must have, therefore,

(3.6) for each $v \in V$ there exists $x \in M$ such that $x(\delta(v)) = 0$.

Now we show that

(3.7) every $x \in M$ satisfies $x(E^+) = |V(G^+)| - 1$

which will mean that every $x \in M$ induces a nn-matching of G^+.

Suppose that some $x \in M$ satisfies $x(\delta(u) \cap E^+) = x(\delta(v) \cap E^+) = 0$ for some $u, v \in V(G^+)$. Assume that x, u and v are chosen so that the distance in G^+ from u to v is minimum. (This is well-defined in view of (3.5).) If u and v were adjacent in G^+, then by defining $x_j \equiv 2$ for an edge of E^+ joining u and v, we would violate the validity of the constraint $ax \leq \alpha$. There therefore exists a node w on a shortest path in G^+ from u to v and by (3.6) we can find $\hat{x} \in M$ satisfying $\hat{x}(\delta(w)) = 0$.

If, starting from w, we follow the path in G^+, consisting alternately of edges with $x_j = 2$ and $\hat{x}_j = 2$, and replace the values of x_j with those of $\hat{x}_j$ for the edges in the path, then we will obtain a $\{0,2\}$-matching $x^* \in M$ deficient at w and (at least) one of u,v. But this contradicts the minimal distance property of x,u,v. Thus $|V(G^+)|$ is odd and (3.7) is established.

If $T = \emptyset$, or if there exists a basis B of $M \cup T$ which is contained in M, then $ax \leq \alpha$ must be the constraint $x(E^+) \leq |V(G)| - 1$ since, by (3.7), every element in the basis satisfies it as an equality. This is a 0,1 facet and thus it must be of the type (2.11) or (2.12) by Theorem 2.12. Since we assume $ax \leq \alpha$. is not a 0-1-constraint, therefore, every basis of $M \cup T$ contains at least one tour.

(We remark that to this point this proof parallels a proof of Lovász [12], who gave a nonalgorithmic proof of the sufficiency of the linear system (2.6)-(2.8) for the 1-matching polytope.)

So we must have $T \neq \emptyset$. We will show

(3.8) for any tour $t \in T$, either $E(t) \subseteq E^+$ or else $E(t)$ induces a hamilton path in G^+.

For suppose $E(t) \not\subseteq E^+$. Then there is $j \in E(t)$ with $a_j = 0$ and $E(t) - \{j\}$ consists of the edges of an even length path of G. This path can be expressed as the average of two complementary np-matchings x^1 and x^2 of G, and $at = .5\, ax^1 + .5\, ax^2 = \alpha$ which implies that $x^1, x^2 \in M$ (since every $x \in P(G)$ satisfies $ax \leq \alpha$). If $E(t) \cap E^+$ is not a hamilton path of G^+, then it is easily verified that one of x^1, x^2 will violate (3.7). Thus (3.8) is established.

Now in order to complete the proof, we must show that

(3.9) there exists a basis B^* of $T \cup M$ such that every $t \in B^*$ satisfies $|E(t) - E^+| \leq 1$.

This will imply that G^+ is a spanning subgraph of G, and hence (3.7) will imply (3.2), which combined with (3.6) will prove (3.3).

Let B be a basis of $M \cup T$ containing a minimum number of tours t for which $|E(t) - E^+| > 1$, and suppose that $\bar{t}$ is such a tour. Then $E(\bar{t})$ induces an (even length) hamilton path π^0 of G^+, amd $E(\bar{t}) - E^+$ consists of a single odd length path π^1, which contains an even number of nodes (including the end points u,v which are nodes of G^+). See Figure 3.1.

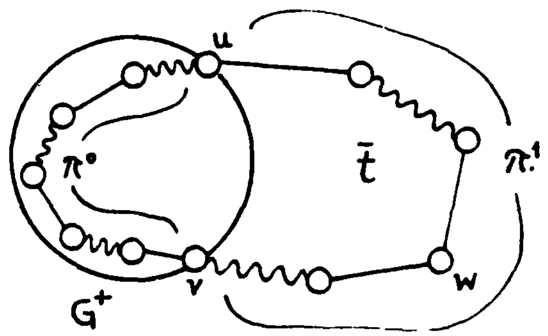

Figure 3.1

Moreover, since $|E(\bar{t}) - E^+| > 1$, we have $|E(\pi^1)| \geq 3$. Now for each $w \in V^* \equiv V(\pi^1) - \{u,v\}$, let x^w be the np-matching of G obtained by taking the (unique) perfect matching of the path obtained from $\bar{t}$ by deleting w.

Let $s \in V(G^+) - \{u,v\}$ and let $\hat{x} \in M$ satisfy $\hat{x}(\delta(s)) = 0$ and $\hat{x}_j = 0$ for all $j \notin E^+$. (By (3.6), there exists $\hat{x} \in M$ satisfying the first property, and we can simply require $\hat{x}_j \equiv 0$ for all $j \in E - E^+$.) Finally, let $\tilde{x}$ be obtained from $\hat{x}$ by giving $\tilde{x}_j$ the value two for the second,fourth,etc. edges of π^1. Then $\tilde{x} \in M$. Now it can be easily verified that

$$|V^*|\bar{t} = \Sigma\ (x^w : w \in V^*) + \tilde{x} - \hat{x},$$

so $\bar{t}$ is a linear combination of $M^+ \equiv \{ x^w : w \in V^* \} \cup \{ \tilde{x}, \hat{x} \}$. Moreover, $\alpha = a\bar{t} = (1/|V^*|)\ \Sigma\ (ax^w : w \in V^*) + \alpha - \alpha$ so we must have $\Sigma\ (ax^w : w \in V^*) = |V^*|\ \alpha$. But since every x^w satisfies $ax^w \leq \alpha$, we must have, therefore, $ax^w = \alpha$ for all $w \in V^*$. Therefore $M^{t} \subseteq M$, and so any basis of $B - \{\bar{t}\} \cup M^t$ will be a basis of $M \cup T$ which contradicts our choice of B. Thus (3.9) is established.

Finally, note that if there existed a basis B of $M \cup T$ such that every tour $t \in B$ satisfied $|E(t) - E^+| = 1$, then every $x \in B$ would satisfy $x(E^+) = |V| - 1$,so our constraint would necessarily be the inequality $x(E^+) \leq |V| - 1$ which induces a 0-1-facet of the form (2.12). Thus (3.4) is established and the proof is complete.

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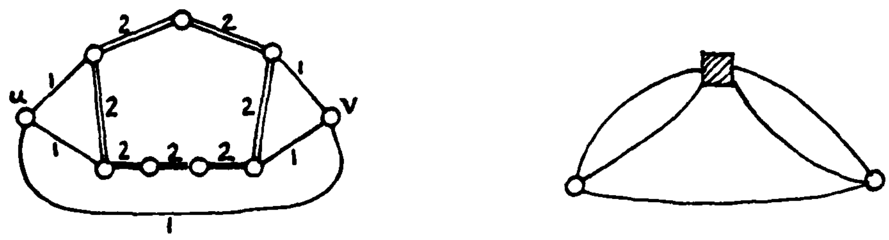
The consequences of this theorem are quite important. It shows that any facet inducing non-0-1-inequality must come from a (spanning) subgraph of G which contains hamilton cycles of G. We now examine such a class of facets.

A simple example of a graph G for which $P(G)$ has a non 0-1 facet is the graph of Figure 3.2 (a). If we let $a = (a_j : j \in E)$ be the vector of edge coefficients indicated in the figure, then $ax \geq 14$ a facet of $P(G)$.



G (a)

G' (b)

Figure 3.2 Graph G for which $P(G)$ has a non 0-1 facet.

It is easy to verify that the inequality is valid; we show that it is a facet by exhibiting $|E(G)| = 12$ affinely independent members of $P(G)$ satisfying $ax = 14$. These will consist of eleven $\{0,2\}$-matchings and one tour. In order to obtain the $\{0,2\}$-matchings, we consider the seven np-$\{0,2\}$-matchings of the centre heptagon and extend each to a np-$\{0,2\}$-matching of G by setting $x_j = 2$ for the edge j joining u and v. These are easily seen to be independent and use only the edge j of the graph G' obtained by contracting the heptagon. (See Fig. 3.2 b). We can give any one of the other four edges of G' the value two and extend it to a np-$\{0,2\}$-matching of G which is near perfect on the heptagon, and thereby obtain four more. Finally, the unique tour in G is affinely independent from the $\{0,2\}$-matchings and so we are done.

Notice that the idea of the construction was to take a large set of np-$\{0,2\}$-matchings which were also near perfect on a certain induced subgraph, and then complete them with a tour. This provides the basis for a general construction.

Let $G' = (V',E')$ be a subgraph of a hamiltonian graph G. We define

$$\tau (G') \equiv |V'| - \max \{ t(E') : t \text{ is a tour of } G \}.$$

We call τ the <u>segment number</u> of G'; it equals the smallest number of segments of some hamilton cycle of G which cover all the nodes of G'. For example, if H is the heptagon of Figure 3.2 a, then $\tau(H) = 2$. If G is non hamiltonian, then the function τ is not defined.

For any $S \subseteq V$ for a graph $G = (V,E)$ we let $G \times S$ denote the graph obtained by contracting the subgraph $G[S]$ to a single pseudonode. Thus, in Figure 3.2, $G' \equiv G \times V(H)$.

Theorem 3.2 . Let $G = (V,E)$ be hamiltonian, let $G' = (V',E')$ be a node induced subgraph of G and suppose that

(3.10) *G' is hypomatchable and nonseparable*

(3.11) *$G \times V'$ is hypomatchable and nonseparable.*

Let $a = (a_j : j \in E)$ be defined by

$$a_j \equiv \begin{cases} \tau(G') & \text{for } j \in E' \\ \tau(G') - 1 & \text{for } j \in E - E' \end{cases}$$

and let

$$\alpha \equiv (\tau(G') - 1)(|V| - 1) + |V'| - 1.$$

Then $ax \leq \alpha$ induces a facet of $P(G)$.

Proof. We first proof the validity of $ax \leq \alpha$. If $\bar{x}$ is a $\{0,2\}$-matching of G, then $\bar{x}(E) \leq |V| - 1$ and $\bar{x}(E') \leq |V'| - 1$ and $a\bar{x} \leq \alpha$. Moreover $a\bar{x} = \alpha$ if and only if $\bar{x}$ is near perfect on both G and G'. If $\bar{x}$ is a tour of G, then $\bar{x}(E) = |V|$ and $\bar{x}(E') \leq |V'| - \tau(G')$ so $a\bar{x} = \tau(G')\bar{x}(E') + (\tau(G')-1)\bar{x}(E-E') \leq (\tau(G')-1)(|V|-1) + |V'|-1 = \alpha$, and we have $a\bar{x} = \alpha$ if and only if $\bar{x}(E') = |V'| - \tau(G')$. Since $ax \leq \alpha$ is valid for all vertices of $P(G)$, it is valid for $P(G)$.

We show that $ax \leq \alpha$ is facet inducing by exhibiting $|E|$ affinely independent members x of $P(G)$ satisfying $ax = \alpha$. By (3.10) and Proposition 2.4, there are $|E'|$ affinely independent np-$\{0,2\}$-matchings of G'. Let $\hat{x}$ be any np-$\{0,2\}$-matching of $G \times V'$ which is deficient at the pseudonode V'. We extend each of our np-$\{0,2\}$-matchings of G' to a np-$\{0,2\}$-matching of G by defining it equal to $\hat{x}$ on $E - E'$. Let X^o be the set of affinely independent matchings thereby obtained. Then $ax = \alpha$ for all $x \in X^o$.

By (3.11) and Proposition 2.4 there are $|E - E'|$ affinely independent np-$\{0,2\}$-matchings of $G \times V'$. Let $\bar{X}$ be such a set, which contains $\hat{x}$, and let X^1 be obtained from $X - \{\hat{x}\}$ by extending each $x \in \bar{X} - \{\hat{x}\}$ to a np-$\{0,2\}$-matching of G. Then X' is a set of $|E - E'| - 1$ affinely independent $\{0,2\}$-matchings of G, each $\chi \in X^1$ satisfies $ax = \alpha$, and it is straightforward to verify that $X^o \cup X^1$ is affinely independent.

Finally, let t be a tour of G satisfying $t(E') = \tau(G')$. Then $at = \alpha$, and since $t(E) = |V|$ but $x(E) = |V| - 1$ for all $x \in X^o \cup X^1$, we see that $X^o \cup X^1 \cup \{t\}$ is affinely independent completing the proof.

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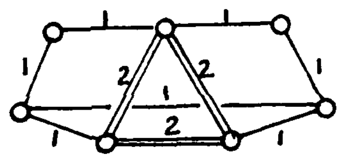
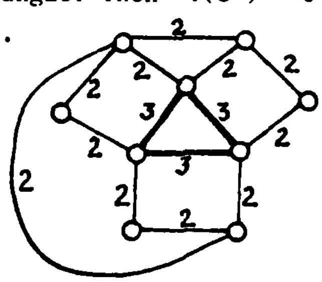
Thus, the example of Figure 3.2 is simply an application of Theorem 3.2, taking G' to be the heptagon. The smallest grapg G we know for which $P(G)$ has such a facet inducing non-(0-1)-inequality is the example of Figure 3.3. We let G' be the triangle, and then $x(E - E') + 2x(E') \leq 8$ is facet-inducing.



righthand side = 8

Figure 3.3 Seven node graph for which $P(G)$ cannot be described by a set of (0-1)-inequalities.

The graph of Figure 3.4 is a nine node example of a graph for which $P(G)$ has a facet-inducing inequality with coefficients 2 and 3. Again G' is the centet triangle. Then $\tau(G') = 3$ and $2x(E - E') + 3x(E') \leq 18$ is facet inducing.



righthand side = 18

Fig. 3.4

We obtain a facet inducing inequality of $P(G)$ for a graph G containing any desired consecutive pair $(s,s+1)$ of integers as its nonzero coefficients by a generalization of the construction of Figure 3.4. Start with an odd polygon P having k nodes, for $k \geq s+1$. Then attach $s+1$ " ears " - paths of length three - to adjacent pairs of nodes of P. Finally choose some node v^* which is an interior node of some ear. Join v^* to the non-corresponding node of each other ear. Then for this graph G it will follow from Theorem 3.2 that

$$(s+1)\, x(E(P)) + s\, x(E - E(P)) \leq sk + 2s^2 + s + k-1$$ is facet inducing for $P(G)$. See Figure 3.5.

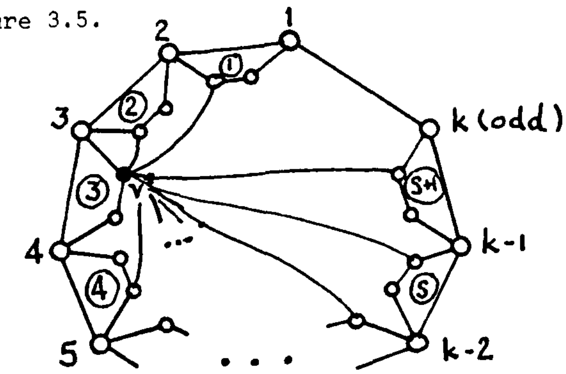


Figure 3.5

Suppose we have a spanning subgraph $G' = (V,E')$ of G and suppose we know a facet inducing inequality $a'x \leq \alpha'$ for $P(G')$. We will say that a facet inducing inequality $ax \leq \alpha$ for $P(G)$ is obtained by lifting $a'x \leq \alpha'$ if

$$\alpha' = \alpha$$
$$a'_j = a_j \quad \text{for all } j \in E'.$$

In other words, we do not change the existing coefficients or righthand side, we simply define those not previously defined in such a manner that the resulting inequality induces a facet of $P(G)$.

A simple method of obtaining such inequalities is the following sequential lifting procedure.

Procedure 3.3 [Sequential Lifting]

Input : $G = (V,E)$, a spanning subgraph $G' = (V,E')$ and a facet inducing inequality $a'x \leq \alpha$ of $P(G')$.

Output : A facet inducing inequality $ax \leq \alpha$ of $P(G)$ obtained by lifting $a'x \leq \alpha$.

Procedure

[0] Initially, define $a_j \equiv a_j'$ for $j \in E'$ and let $S \equiv E'$. S is the set of edges for which a_j has been defined.

[1] For each $j \in E - S$, do the following

[1a] Let u,v be the ends of j. Let

$$a_j \equiv \min \{1/2\,(\alpha - ax) : x \text{ is a } \{0,2\}\text{-matching of the graph } (V,S)[V-\{u,v\}]\}$$
$$\cup\ \{\alpha - ax : x \text{ is the incidence vector of a hamilton path in } (V,S) \text{ from } u \text{ to } v\}.$$

[1b] Let $S \equiv S \cup \{j\}$.

End

Notice that sequential lifting leaves all old coefficients and the righthand side of the inequality unchanged. The idea is to (sequentially) define each a_j as large as possible such that the inequality will remain valid, considering edges in $S \cup \{j\}$. Further, suppose we have a set X' of $|E'|$ affinely indpendent members x of $P(G')$ satisfying $ax = \alpha$. We can enlarge it to such a set X for $ax \leq \alpha$ and $P(G)$ by adding the following step:

[1c] For each $x \in X'$, add a new component $x_j \equiv 0$. If the minimum in [1a] was achieved by a $\{0,2\}$-matching, let $\hat{x}^j$ be this $\{0,2\}$-matching extended by defining $\hat{x}^j_j \equiv 2$. If the minimum was achieved by a hamilton path, let $\hat{x}^j$ be the tour obtained by defining $\hat{x}^j_j \equiv 1$. Let $X' \equiv X' \cup \{\hat{x}^j\}$.

Then if we let X be the final X', the " triangular structure " of the $\hat{x}^j$ will assure that X is affinely independent. This verifies that $ax \leq \alpha$ is indeed facet inducing for $P(G)$. This means, of course, that we will always finish with $a \geq 0$. (See Theorem 3.1).

Sequential lifting can be applied in many different orders to the edges, generally resulting in different lifted inequalities. Moreover, there can be facet inducing inequalities of $P(G)$ obtained by lifting from $a'x \leq \alpha'$, but not obtainable by sequental lifting.

Our main interest in this procedure is that it shows that "unpleasant " facets are, in effect, retained when edges are added to the graph. In particular, if we were to restrict our attention to complete graphs, Theorem 3.2 and Procedure 3.3 show that for n sufficiently large, there is a facet inducing inequality containing any desired consecutive pair of positive integers among the coefficients.

Finally, we can see, using Theorem 3.1, that the new coefficients defined by sequential lifting will never be larger than the largest previously existing coefficient, and generally, these new coefficients tend to decrease to zero as more edges are added, until some constant value is obtained. We conjecture the following :

Conjecture 3.4 *Let K_n be the complete graph on n nodes. For any positive integer s, there exists an integer $N(s)$ such that for $n \leq N(s)$, there is a facet inducing inequality of $P(K_n)$ whose coefficients include all integers from 0 to s.*

4. Facets of the Travelling Salesman Polytope

In this section we discuss the relationship between facets of TSP(G) and facets of P(G). Most studies of TSP(G) are restricted to the case of G being a complete graph, because solving travelling salesman problems on complete graphs is polynomially equivalent to the more general problem. An interesting feature of the results of the previous sections is that they do apply to general graphs. However, in this section we too will restrict ourselves to complete graphs to facilitate comparison with previously known results. We adopt the notation of Grötschel and Padberg [10] and let Q_T^n denote $TSP(K_n)$. We let E_n and V_n respectively denote the edge set and node set of K_n. First we mention two preliminary results.

Proposition 4.1. ([10] Theorem 2.2.) *The dimension of* Q_T^n *is* $n(n-3)/2 = |E_n| - |V_n|$ *for* $n \geq 3$.

Corollary 4.2. *The minimal affine space containing* Q_T^n *is equal to*

$$\{x \in \mathbb{R}^{E_n} : x(\delta(i)) = 2 \quad \text{for all} \quad i \in V_n\}$$

The importance of this corollary is that it completely characterizes which inequalities induce the same facet as some presribed facet inducing inequality for Q_T^n. We summarize this as follows.

Corollary 4.3. *Let* $ax \geq \alpha$ *be a valid inequality for* Q_T^n. *Then for any* $\lambda = (\lambda_i \in \mathbb{R} : i \in V_n)$ *and any* $\mu > 0$, *the inequality*

$$(\mu a)x + \sum_{i \in V_n} \lambda_i x(\delta(i)) \leq \mu\alpha + 2\lambda(V_n) \tag{4.1}$$

is a valid inequality for Q_T^n. *Moreover, if we let* $F \equiv \{x \in Q_T^n : ax = \alpha\}$, *then the set of members of* Q_T^n *satisfying (4.1) with equality is* F. *If* F *is nonempty, then every inequality whose corresponding hyperplane intersects* Q_T^n *in exactly* F *is of the form (4.1) for appropriate* μ *and* λ.

Of course, this corollary is a specialization of a fundamental polyhedral result: The inequalities in a linear system that defines a polyhedron are only unique up to positive multiples and the addition of equations satisfied by all members of the polyhedron.

For the remainder of this section, we let $P_n \equiv P(K_n)$, for $n \geq 3$. We now prove a basic result relating the facets of Q_T^n and P_n for $n \geq 3$, odd.

It states that for each facet F of Q_T^n, there exists a unique facet F' of P_n such that $F = F' \cap Q_T^n$. This uniqueness is not true for general polyhedra, as illustrated in Figure 4.1. In each case the "ridge pole" is a face of the "tent" and the marked end of the ridge pole is a facet of the face. Figure 4.1a has the uniqueness property, but Figure 4.1b does not.

Figure 4.1

Finally, we remark that the proof of the following theorem will consist of an algorithm which starts with a facet defining inequality $ax \leq \alpha$ of Q_T^n and transforms it into a facet defining inequality of P_n, which defines the same facet of Q_T^n.

Theorem 4.4. *For any facet F of Q_T^n, there exists a unique facet F' of P_n such that $F = F' \cap Q_T^n$.*

Proof. Let F be a facet of Q_T^n and suppose that $F = \{x \in Q_T^n : ax = \alpha\}$ where $ax \leq \alpha$ is a valid inequality for Q_T^n. We assume that $a \geq 0$. If not we add sufficiently high multiples of degree constraints so as to have this property. If we consider the inequality

$$(4.2) \qquad ax + \sum_{i \in V} \lambda_i \, x(\delta(i)) \leq \alpha + 2 \sum_{i \in V} \lambda_i$$

we can see that varying λ_i for a node i has no effect on the feasibility of a $\{0,2\}$ matching $\bar{x}$ satisfying $\bar{x}(\delta(i)) = 2$ or on a tour $\bar{x}$ which also must satisfy $\bar{x}(\delta(i)) = 2$. However, if $\bar{x}$ is a $\{0,2\}$ matching deficient at i, then by choosing an appropriate value from λ_i, we can ensure

(4.3) every $\{0,2\}$ matching deficient at i satisfies (4.2),

(4.4) there exists an np - $\{0,2\}$ - matching deficient at i which satisfies (4.2) with equality.

In fact it can be verified that this is given by

(4.5) $\lambda_i \equiv 1/2 \max\{ax - \alpha : x$ is a $\{0,2\}$-matching of G deficient at $i\}$.

Since $a \geq 0$, this maximum will always be attained for a near perfect matching. Since the choice of deficient node provides a partition of the near perfect matchings, the λ_i are determined independently and uniquely. Thus, using Corollary 4.3, there is a unique (up to a positive multiple) inequality that is valid for Q^n_T and P_n, induces F and satisfies (4.3) and (4.4), two necessary conditions for it to be facet inducing for P_n. This is (4.2) with λ_i defined as in (4.5). Let $F' \equiv \{x \in P_n : x$ satisfies (4.2) with equality, for λ_i as in (4.5)$\}$. We show that F' is a facet of P_n.

Since F is a facet of Q^n_T , there exists a set T of $|E_n| - |V_n|$ affinely independent tours satisfying (4.2) with equality. For each node i , our choice of λ_i ensures that there exists a np-$\{0,2\}$-matching x^i deficient at i satisfying (4.2) with equality. Let $M \equiv \{x^i : i \in V_n\}$. Then for each $i \in V_n$, x^i is the only member of $T \cup M$ which does not satisfy $x(\delta(i)) = 2$. Thus it is affinely independent from $T \cup M - \{x^i\}$. Therefore $T \cup M$ is affinely independent of cardinality $|E_n|$ so F' is a facet of P_n and $F' \cap P_n = F$. $\square$

Perhaps surprisingly, there are presently only three classes of facets for Q^n_T appearing in the literature. The first such class, the so called "trivial" facets,are those induced by nonnegativity constraints $x_j \geq 0$ for all $j \in E_n$ ([10] Theorem 3.2). These obviously correspond to the inequalitities (2.9) for P_n .

The second class of facets are those induced by the subtour elimination constraints $x(\gamma(S)) \leq |S| - 1$ for $S \subseteq V_n$, $2 \leq |S| \leq n-1$ ([10] Theorem 6.1). For any such S , the subtour elimination constraints corresponding to S and $V_n - S$ induce the same facet of Q^n_T . (Simply sum one half the degree constraints for all nodes in $V_n - S$, subtract one half the sum of degree constraints for nodes in S and add this to the constraint $x(\gamma(S)) \leq |S| - 1$.) In particular, the edge "capacity" constraints $x_j \leq 1$ for $j \in E$ induce the same facets as the subtour elimination constraints for the cardinality $n-2$ subsets of V_n . By Theorem 4.4. there exists a unique facet of P_n which determines this "doubly defined" facet of Q^n_T . Of course, this is the inequality (2.11) for the odd cardinality one of $|S|$, $|V_n - S|$.

The third class of facets, induced by generalized comb constraints is more complex. Let $W_i \subseteq V_n$ for $i = 0,1,\ldots,k$ satisfy

(4.6) $|W_o \cap W_i| \geq 1$ for $i=1,2,\ldots,k$

(4.7) $|W_i - W_o| \geq 1$ for $i=1,2,\ldots,k$

(4.8) $$|W_i \cap W_j| = 0 \quad \text{for} \quad 1 \le i < j \le k$$

Then we call the graph C with nodeset $\bigcup_{i=0}^{k} W_i$ and edge set $\bigcup_{i=0}^{k} \gamma(W_i)$ a comb in K_n ; W_o is the handle and W_i are called the teeth for $i=1,\dots,k$. The comb inequality corresponding to C is given by

$$\sum_{i=0}^{k} x(r(W_i)) \le |W_o| + \sum_{i=1}^{k} (|W_i| - 1) - \lceil k/2 \rceil$$

where for $r \in \mathbb{R}$, "$\lceil r \rceil$" denotes the smallest integer no less than r .

Note that the coefficients of a comb inequality will be 0,1 or 2. Such inequalities were introduced by Chvátal [1] who required equality in (4.6), resulting in a 0 - 1 - inequality. We call such a comb simple. In a simple comb, each tooth has exactly one node in the handle. In a general comb, a tooth may have several nodes in the handle, and all edges joining these nodes have coefficient two in the inequality. A major result of Grötschel and Padberg ([10] Theorem 6.2) is that for $k \ge 3$, odd, every comb inequality induces a facet of Q_T^n .

It is a routine matter to apply the procedure of the proof of Theorem 4.4 in order to find the corresponding facet inducing inequality for P_n . We illustrate this with the following.

Theorem 4.5. *Let C be a simple comb having an odd number k of teeth such that $|V(C)|$ is even. If $|V(C)| \le n - \lceil k/2 \rceil$, then the facet of P_n corresponding to the facet of Q_T^n , induced by the comb inequality for C is obtained by sequentially lifting a facet of the form of Theorem 3.2 for a subgraph G of K_n .*

Proof. Let $ax \le \alpha$ be the comb inequality corresponding to C . Let G be the spanning subgraph of K_n whose edge set consists of $E(C)$ together with those edges having at least one end not in the comb. That is, we exclude those edges both of whose ends are in $V(C)$, but which are not in $E(C)$. We will show first that an application of the procedure of the proof of Theorem 4.4 obtains a facet inducing inequality for $P(G)$ of the form of Theorem 3.2.

We first compute the value λ_i for each $i \in V_n$, as given by (4.5). For $i \in V(C)$, the maximum value of ax for a $\{0,2\}$ - matching x deficient at i is $|V(C)| - 2$. For $i \in V_n - V(C)$, this maximum is $|V(C)|$. Therefore

$$\lambda_i = \begin{cases} 1/2 \lceil k/2 \rceil - 1 & \text{for} \quad i \in V(C) \\ 1/2 \lceil k/2 \rceil & \text{for} \quad i \in V_n - V(C) \quad . \end{cases}$$

When we use these values of λ_i in (4.2), we obtain the following new coefficients a_j' for each edge j.

$$a_j' = \begin{cases} \lceil k/2 \rceil - 1 & \text{for} \quad j \in E(C) \\ \lceil k/2 \rceil - 1 & \text{for} \quad j \in \delta(V(C)) \\ \lceil k/2 \rceil & \text{for} \quad j \in \gamma(V_n - V(C)) \quad . \end{cases}$$

Now let $S \equiv V_n - V(C)$. Then $G[S]$ is a complete graph on an odd number of nodes, and so is hypomatchable and nonseparable. The graph $G \times S$ can also be easily checked to be hypomatchable and nonseparable. Because $|S| = n - |V(C)| \geq \lceil k/2 \rceil$, G is hamiltonian. Moreover, the segment number $\tau(G[S]) = \tau(C) = \lceil k/2 \rceil$ so the coefficients a_j' are in fact given by

$$a_j' = \begin{cases} \tau(G[S]) & \text{for} \quad j \in \gamma(S) \\ \tau(G[S]) - 1 & \text{for} \quad j \in E(G) - \gamma(S) \; . \end{cases}$$

These then are the coefficients of a facet of the form of Theorem 3.2 for G, so the resulting right hand side, α', must equal

$$(\tau(G[S]) - 1)(n - 1) + |S| - 1$$

as prescribed by the theorem.

Now let $a''x \leq \alpha''$ be the facet of P_n corresponding to the comb inequality for C. By again using the procedure of the proof of Theorem 4.4, we see that

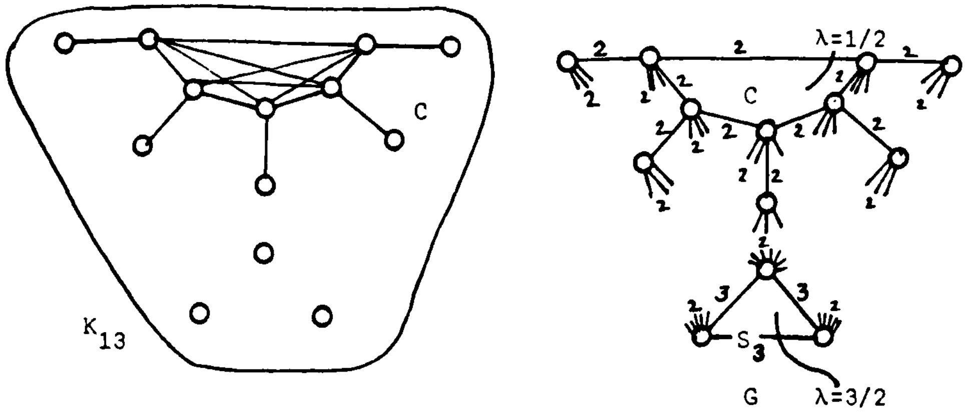
$$a_j'' = \begin{cases} a_j'' & \text{for} \quad j \in E(G) \\ \tau(C[S]) - 2 & \text{for} \quad j \in E_n - E(G) \end{cases}$$

$$\alpha'' = \alpha' \quad .$$

We will complete the proof by proving that the values $a_j'' = \tau(C[S]) - 2$ for $j \in E_n - E(G)$ are those given by sequential lifting.

Suppose that we have sequentially lifted the coefficients for the edges of some (possibly empty) subset of $E_n - E(G)$ and obtained the desired value. Let $j \in E_n - (J \cup E(G))$. Let u,v be the ends of j, let G^j denote the graph $(V_n, J \cup E(G))$, and let $\overline{G}^j$ denote $G^j[V_n - \{u,v\}]$. Then the maximum value of a $\{0,2\}$-matching of $\overline{G}^j$ is $\alpha' - 2(\tau(G[S]) - 1)$. The maximum sum of the edge costs of a hamilton path in G^j from u to v is $\alpha' - (\tau(G[S]) - 2)$. Thus sequential lifting will define $a''_j \equiv \tau(G[S]) - 2)$ and the proof now follows by induction. □

Figure 4.2 illustrates a small example of this process. Let C be the ten node-five tooth comb of

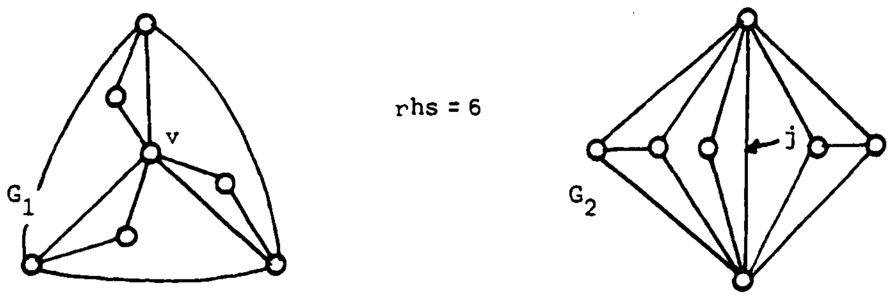


(a) (b)

Figure 4.2

Figure 4.2(a), in K_{13}. The comb inequality gives each edge the coefficient 1 and has $\alpha = 7$. The graph G of Figure 4.2(b) consists of C , the subgraph induced by the set S of non-comb nodes and all edges joining these two parts. The procedure of Theorem 4.4 calculates $\lambda_i = 1/2$ for the nodes i of C and $\lambda_i = 3/2$ for the nodes i of S . Thus the coefficients a'_j are as indicated, three for edges of $\gamma(S)$ and two for all other edges and $\alpha' = 26$. Sequential lifting will then cause all edges of $E_{13} - E(G)$ to have the coefficient one, which induces the facet of P_{13} corresponding to the facet of Q_T^{13} induced by C .

It is certainly possible to study the results of applying the procedure of Theorem 4.4 to combs having an odd number of nodes. In fact, this can be used to provide other classes of non - 0 - 1 - facets of P_n . However, for the remainder of this section we wish to briefly discuss cases when facet inducing inequalities of P_n induce facets of Q_T^n . In particular, for inequalities of the form (2.12.), those induced by hypomatchable nonhamiltonian nonseparable edge maximal subgraphs. A graph G is sai to be <u>hypohamiltonian</u> if G is nonhamiltonian, but $G - \{v\}$ is hamiltonian for all $v \in V$. It is an easy exercise to verify that if n is odd, then any edge maximal spanning hypohamiltonian subgraph of K_n satisfies our conditions of (2.12). Grötschel [8] showed that those spanning edge-maximal hypohamiltonian subgraphs of K_n which satisfy a certain technical property, do induce a facet of the monotone travelling salesman polytope. (He did not settle whether or not this technical property was indeed necessary.) Thus there is an obvious correction between our inequalities (2.12) and the monotone polytope. For the travelling salesman polytope itself, some inequalities (2.12) are facet inducing and some are not. For example the inequality (2.12) for the graph G_1 of Figure 4.3(a) is facet inducing for Q_T^7 , but that of the graph G_2 of Figure 4.3(b) is not. (Note that both graphs satisfy the necessary conditions for (2.12) to apply, but neither is hypohamiltonian.)



(a) Facet inducing for Q_T^7

(b) Not facet inducing for Q_T^7

Figure 4.3

The inequality (2.12) for G_1 is equivalent to the facet inducing inequality of the comb obtained by deleting node v . On the other hand, edge j of G_2 belongs to no hamilton cycle of K_7 which contains six edges of G_2 . Therefore the inequality (2.12) for Q_T^7 is implied by the nonnegativity constraint for edge j .

However, there are examples of inequalities of the form (2.12) which are facet inducing for Q_T^n and which do not seem to arise from any known class of facet inducing inequalities. For example, the inequality (2.12) for the modified Petersen graph of Figure 4.4 is facet inducing for K_{11} .

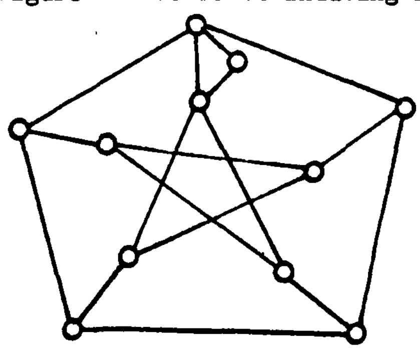


Figure 4.4 Modified Petersen Graph

This can be shown using a slight modification of the proof of Maurras [13] that the inequality $x(E) \leq 9$ is facet inducing for Q_T^{10}, where E is the edge set of any subgraph which is a Peterson graph.

5. Concluding Remarks

When we wish to study a polytope such as the travelling salesman polytope, which is not of full dimension, we generally have considerable choice as to which full dimension polyhedron (if any) we will embed it in. We have studied, here, a particular polyhedron, P_n, which has several interesting properties. First (Theorem 4.4), for any facet of Q_T^n , there is a unique facet of P_n which intersects Q_T^n in exactly this facet. Thus we can partition the facets of P_n into three classes : those that contain all of Q_T^n , those that contain no facet of Q_T^n , and those that intersect Q_T^n in a facet. Theorem 4.4 shows that there is a bijection between the facets in the third class and the facets of Q_T^n.

In Theorem 2.12 we completely characterized those facets of $P(G)$, for general G, for which the inducing inequality can be scaled so as to have 0-1 valued coefficients. The most interesting set of facet inducing 0-1-inequalities were those of (2.12). At the end of the previous section we saw that some of these do indeed induce facets of TSP(G) itself.

In Theorem 3.1, we determined several properties possessed by non-0-1 inequalities which induce facets of P(G). One of these properties is that the subgraph of G induced by the edges having positive coefficients in such an inequality must be spanning, and indeed, must be hamiltonian. This has one rather negative consequence : Such inequalities will probably be harder to use in a cutting plane approach than, for example, the comb inequalities which have been used so successfully by Grötschel [9] to solve a " real world " travelling salesman problem. However, a possible area for future research would be to see if " simpler " equivalent inequalities (for TSP(G)) can be found for classes of such inequalities.

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References

1. V. Chvátal, "Edmonds polytopes and weakly Hamiltonian graphs", Mathematical Programming 5 (1973) 29-40.
2. G. B. Dantzig, D. R. Fulkerson and S. M. Johnson, "Solution of a large-scale travelling salesman problem", Operations Research 2 (1954) 393-410
3. J. Edmonds, "Maximum matching and a polyhedron with 0,1 vertices", Journal of Research of the National Bureau of Standards Sect. B 69 (1965) 125-130.
4. J. Edmonds, "Matroids and the greedy algorithm", Mathematical Programming 1 (1971) 127-136.
5. J. Edmonds, "Matroid Intersection", Annals of Discrete Mathematics 4 (1979) 39-49
6. J. Edmonds, "Optimum branchings", in: G. B. Dantzig, A. F. Veinott (eds.): Mathematics of the Decision Sciences, Part 1, American Mathematical Society, Providence (1968) pp 346-361.
7. M. Grötschel, Polyedrische Characterisierungen kombinatorischer Optimieringsprobleme, Dissertation, Universität Bonn, 1977 (Verlag A. Hain, Meisenheim, 1977).
8. M. Grötschel, "On the monotone symmetric travelling salesman problem: hypohamiltonian / hypotraceable graphs and facets", Mathematics of Operations Research 5 (1980) 285-292.
9. M. Grötschel, "On the symmetric travelling salesman problem: solution of a 120-city problem", Mathematical Programming Studies 12 (1980) 61-77.
10. M. Grötschel and M. W. Padberg, "On the symmetric travelling salesman problem I,II", Mathematical Programming 16 (1979) 265-302.

11. L. Lovász, " A note on factor-critical graphs", Studia Scientiarum Mathematicarum Hungarica 7 (1972) 279-280.

12. L. Lovász, "Graph theory and integer programming", Annals of Discrete Mathematics 4 (1979) 141-158.

13. J. F. Maurras, "Polytopes à sommets dans $\{0,1\}^n$", dissertation Univ. Paris VII (1976).

14. M. W. Padberg and S. Hong, "On the symmetric travelling salesman problem: A computational study", T. J. Watson Research Report, IBM Research (Yorktown Heights, 1977).

15. W. R. Pulleyblank and J. Edmonds, "Facets of 1-matching polyhedra" in C. Berge and D. K. Ray.Chandhuri eds. Hypergraph Seminar Springer Verlag (1974), 214-242.

16. T. W. Tutte, "The factorization of linear graphs", Journal of the London Mathematical Society 22 (1947) 107-111.

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A (0,2)-matching is an assignment of the integers 0,2 to the edges of a graph G such that for every node the sum of the integers on the incident edges is at most two. A tour is the 0-1-incidence vector of a hamilton cycle. We study the polytope P(G), defined to be convex hull of the (0,2)-matchings and tours of G. When G has an odd number of nodes, the travelling salesman polytope, the convex hull of the tours, is a facet of P(G). We obtain the following results: (OVER)

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(i) We completely characterize those facets of P(G) which can be induced by an inequality with 0-1-coefficients.

(ii) We prove necessary properties for any other facet inducing inequality, and exhibit a class of such inequalities with the property that for any pair of consecutive positive integers, there exists an inequality in our class whose coefficients include these integers.

(iii) We relate the facets of P(G) to the facets of the travelling salesman polytope. In particular, we show that for any facet F of the travelling salesman polytope, there is a unique facet of P(G) whose intersection with the travelling salesman polytope is exactly F.

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